

# CARATHÉODORY FUNCTIONS IN THE BANACH SPACE SETTING

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ABSTRACT. We prove representation theorems for Carathéodory functions in the setting of Banach spaces.

## 1. INTRODUCTION

L. de Branges and J. Rovnyak introduced in [7], [8] various families of reproducing kernel Hilbert spaces of functions which take values in a Hilbert space and are analytic in the open unit disk or in the open upper half-plane. These spaces play an important role in operator theory, interpolation theory, inverse scattering, the theory of wide sense stationary stochastic processes and related topics; see for instance [11], [4], [5]. In the case of the open unit disk  $\mathbb{D}$ , of particular importance are the following two kinds of reproducing kernels:

$$(1.1) \quad \begin{aligned} k_\phi(z, w) &= \frac{\phi(z) + \phi(w)^*}{2(1 - zw^*)}, \\ k_s(z, w) &= \frac{I - s(z)s(w)^*}{1 - zw^*}. \end{aligned}$$

In these expressions,  $s(z)$  and  $\phi(z)$  are operator-valued functions analytic in  $\mathbb{D}$ ,  $*$  denotes the Hilbert space adjoint, and  $I$  denotes the identity operator. The functions for which the kernels  $k_\phi(z, w)$  and  $k_s(z, w)$  are positive are called respectively Carathéodory and Schur functions. We remark that one can use the Cayley transform

$$s(z) = (I - \phi(z))(I + \phi(z))^{-1}$$

to reduce the study of the kernels  $k_\phi(z, w)$  to the study of the kernels  $k_s(z, w)$ . For these latter it is well known that the positivity of the kernel  $k_s(z, w)$  implies analyticity of  $s(z)$ .

Every Carathéodory function admits two equivalent representations. The first, called the Riesz – Herglotz representation, reads as follows:

$$(1.2) \quad \phi(z) = ia + \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t)$$

where  $a$  is a real number and where  $\mu(t)$  is an increasing function such that  $\mu(2\pi) < \infty$ . The integral is a Stieltjes integral and the proof relies on Helly's theorem; see [10, pp. 19–27].

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The second representation reads:

$$(1.3) \quad \phi(z) = ia + \Gamma(U + zI)(U - zI)^{-1}\Gamma^*$$

where  $a \in \mathbb{R}$  and where  $U$  is a unitary operator in an auxiliary Hilbert space  $\mathcal{H}$  and  $\Gamma$  is a bounded operator from  $\mathcal{H}$  into  $\mathbb{C}$ .

The expression (1.3) still makes sense in a more general setting when the kernel  $k_\phi(z, w)$  has a finite number of negative squares. The space  $\mathcal{H}$  is then a Pontryagin space. This is the setting in which Kreĭn and Langer proved this result; see [17, Satz 2.2 p. 361]. They allowed the values of the function  $\phi(z)$  to be operators between Pontryagin spaces and required weak continuity at the origin. Without this hypothesis one can find functions for which the kernel  $k_\phi(z, w)$  has a finite number of negative squares but which are not meromorphic in  $\mathbb{D}$  and in particular cannot admit representations of the form (1.3); for instance the function

$$\phi(z) = \begin{cases} 0 & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$$

defines a kernel  $k_\phi(z, w)$  which has one negative square; see [3, p. 82] for an analogue for  $k_s(z, w)$  kernels.

Operator-valued Carathéodory functions were extensively studied in the Hilbert space case; see e.g. [13], [14], [19], [20]. We would also like to mention the non-stationary setting, where an analogue of the representation (1.3) was obtained for upper-triangular operators; see [1] and [2].

The notion of reproducing kernel space (with positive or indefinite metric) can also be introduced for functions which take values in Banach spaces and even topological vector spaces. The positive case was studied already by Pedrick for functions with values in certain topological vector spaces in an unpublished report [21] and studied further by P. Masani in his 1978 paper [18]. Motivations originate from the theory of partial differential equations (see e.g. [12]) and the theory of stochastic processes (see e.g. [15, §4] and [24]).

The present paper is devoted to the study of Carathéodory functions whose values are bounded operators between appropriate Banach spaces. It seems that there are no natural analogs of Schur functions or of the Cayley transform in this setting.

Let  $\mathcal{B}$  be a Banach space. We denote by  $\mathcal{B}^*$  the space of anti-linear bounded functionals (that is, its conjugate dual space). The duality between  $\mathcal{B}$  and  $\mathcal{B}^*$  is denoted by

$$\langle b_*, b \rangle_{\mathcal{B}} \stackrel{\text{def}}{=} b_*(b), \quad \text{where } b \in \mathcal{B} \text{ and } b_* \in \mathcal{B}^*.$$

An  $\mathbf{L}(\mathcal{B}, \mathcal{B}^*)$ -valued function  $\phi(z)$  defined in some open neighborhood  $\Omega$  of the origin and weakly continuous at the origin will be called a Carathéodory function if the  $\mathbf{L}(\mathcal{B}, \mathcal{B}^*)$ -valued kernel

$$(1.4) \quad k_\phi(z, w) = \frac{\phi(z) + \phi(w)^*|_{\mathcal{B}}}{2(1 - zw^*)}$$

is positive in  $\Omega$ . The notion of positivity for bounded operators and kernels from  $\mathcal{B}$  into  $\mathcal{B}^*$  is reviewed in the next section. We shall prove (see Theorem 5.2) that every Carathéodory function admits a representation of the form (1.3) and, in particular, admits an analytic extension to  $\mathbb{D}$ ; see e.g. [22, pp. 189–190] for information on vector-valued analytic functions. We note that the proof of this theorem can be adapted to the case when the kernel  $k_\phi(z, w)$  has a finite number of negative squares; see Remark 5.4.

Furthermore, if  $\mathcal{B}$  is a separable Banach space then  $\mathbf{L}(\mathcal{B}, \mathcal{B}^*)$ -valued Carathéodory functions can be characterized as functions analytic in  $\mathbb{D}$  and such that

$$\phi(z) + \phi(z)^* \big|_{\mathcal{B}} \geq 0, \quad z \in \mathbb{D}.$$

Moreover, in this case we have an analogue of the Riesz – Herglotz representation (1.2); see Theorem 5.5.

We conclude with the outline of the paper; the next three sections are of preliminary nature, and deal with positive operators, Stieltjes integrals and Helly’s theorem respectively. Representation theorems for Carathéodory functions are proved in Section 5. Two cases are to be distinguished, as whether  $\mathcal{B}$  is separable or not. The case of  $\mathbf{L}(\mathcal{B}^*, \mathcal{B})$ -valued Carathéodory functions will be treated in the last section of this paper. This case is of special importance. Indeed, if  $\phi(z)$  is a  $\mathbf{L}(\mathcal{B}^*, \mathcal{B})$ -valued Carathéodory function which takes invertible values, its inverse is a  $\mathbf{L}(\mathcal{B}, \mathcal{B}^*)$ -valued Carathéodory function. In the Hilbert space case, this fact has important connections with operator models for pairs of unitary operators (see [9] and, for the analogue for self-adjoint operators, [7], [6]). We will explore the Banach space generalizations of these results in a future publication.

## 2. POSITIVE OPERATORS AND POSITIVE KERNELS

In this section we review for the convenience of the reader various facts on bounded positive operators from  $\mathcal{B}$  into  $\mathcal{B}^*$ . First some notations and a definition.

**Definition 2.1.** Let  $\mathcal{B}$  be a complex Banach space and let  $A \in \mathcal{L}(\mathcal{B}, \mathcal{B}^*)$ . The operator  $A$  is said to be *positive* if

$$\langle Ab, b \rangle_{\mathcal{B}} \geq 0, \quad \forall b \in \mathcal{B}.$$

Note that a positive operator is in particular self-adjoint in the sense that  $A = A^* \big|_{\mathcal{B}}$ , that is,

$$(2.1) \quad \langle Ab, c \rangle_{\mathcal{B}} = \overline{\langle Ac, b \rangle_{\mathcal{B}}}.$$

Indeed, (2.1) holds for  $b = c$  in view of the positivity. It then holds for all choices of  $b, c \in \mathcal{B}$  by polarization:

$$\begin{aligned} \langle Ab, c \rangle_{\mathcal{B}} &= \frac{1}{4} \sum_{k=0}^3 i^k \langle A(b + i^k c), (b + i^k c) \rangle_{\mathcal{B}} \\ &= \frac{1}{4} \sum_{k=0}^3 i^k \langle A(c + i^{-k} b), (c + i^{-k} b) \rangle_{\mathcal{B}} \\ &= \frac{1}{4} \sum_{k=0}^3 i^{-k} \langle A(c + i^{-k} b), (c + i^{-k} b) \rangle_{\mathcal{B}} \\ &= \overline{\langle Ac, b \rangle_{\mathcal{B}}}. \end{aligned}$$

Now, let  $\tau$  be the natural injection from  $\mathcal{B}$  into  $\mathcal{B}^{**}$ :

$$(2.2) \quad \langle \tau(b), b_* \rangle_{\mathcal{B}^*} = \overline{\langle b_*, b \rangle_{\mathcal{B}}}.$$

We have for  $b, c \in \mathcal{B}$ :

$$\begin{aligned} \langle A^* \tau c, b \rangle_{\mathcal{B}} &= \langle \tau c, Ab \rangle_{\mathcal{B}^*} \\ &= \overline{\langle Ab, c \rangle_{\mathcal{B}}} \\ &= \langle Ac, b \rangle_{\mathcal{B}} \end{aligned}$$

in view of (2.1), and hence  $A = A^*|_{\mathcal{B}}$ .

The following factorization result is well known and originates with the works of Pedrick [21] (in the case of topological vector spaces with appropriate properties) and Vakhania [24, §4.3.2 p.101] (for positive elements in  $\mathbf{L}(\mathcal{B}^*, \mathcal{B})$ ); see the discussion in [18, p. 416]. We refer also to [15] for the case of barreled spaces and to [12] for the case of unbounded operators.

**Theorem 2.2.** *The operator  $A \in \mathcal{L}(\mathcal{B}, \mathcal{B}^*)$  is positive if and only if there exist a Hilbert space  $\mathcal{H}$  and a bounded operator  $T \in \mathbf{L}(\mathcal{B}, \mathcal{H})$  such that  $A = T^*T$ . Moreover,*

$$(2.3) \quad \langle Ab, c \rangle_{\mathcal{B}} = \langle Tb, Tc \rangle_{\mathcal{H}}, \quad b, c \in \mathcal{B}$$

and

$$(2.4) \quad \sup_{\|b\|=1} \langle Ab, b \rangle_{\mathcal{B}} = \|A\| = \|T\|^2.$$

Finally we have

$$(2.5) \quad |\langle Ab, c \rangle_{\mathcal{B}}| \leq \langle Ab, b \rangle_{\mathcal{B}}^{1/2} \langle Ac, c \rangle_{\mathcal{B}}^{1/2}.$$

*Proof.* For  $Ab$  and  $Ac$  in the range of  $A$  the expression

$$(2.6) \quad \langle Ab, Ac \rangle_A = \langle Ab, c \rangle_{\mathcal{B}} = \overline{\langle Ac, b \rangle_{\mathcal{B}}}$$

is well defined in the sense that  $Ab = 0$  (resp.  $Ac = 0$ ) implies that (2.6) is equal to 0. Thus formula (2.6) defines a sesquilinear form on  $\text{ran } A$ . It is positive since the operator  $A$  is positive. Moreover, it is non-degenerate because if  $\langle Ab, b \rangle_{\mathcal{B}} = 0$  then  $Ab = 0$ .

Indeed, if  $c$  is such that  $\langle Ac, c \rangle_{\mathcal{B}} = 0$  then (2.1) implies that the real and the imaginary parts of  $\langle Ab, c \rangle_{\mathcal{B}}$  are equal, respectively, to

$$\frac{1}{2} \langle A(b+c), b+c \rangle_{\mathcal{B}} \quad \text{and} \quad \frac{1}{2} \langle A(b+ic), b+ic \rangle_{\mathcal{B}}$$

and, therefore, are non-negative. Then the same can be said about  $\langle Ab, -c \rangle_{\mathcal{B}}$ , hence

$$(2.7) \quad \langle Ab, c \rangle_{\mathcal{B}} = 0.$$

Furthermore, if  $c$  is such that  $\langle Ac, c \rangle_{\mathcal{B}} > 0$  then we have

$$0 \leq \left\langle A \left( b - \frac{\langle Ab, c \rangle_{\mathcal{B}}}{\langle Ac, c \rangle_{\mathcal{B}}} c \right), b - \frac{\langle Ab, c \rangle_{\mathcal{B}}}{\langle Ac, c \rangle_{\mathcal{B}}} c \right\rangle_{\mathcal{B}} = -\frac{|\langle Ab, c \rangle_{\mathcal{B}}|^2}{\langle Ac, c \rangle_{\mathcal{B}}},$$

hence (2.7) holds in this case, as well.

Thus,  $(\text{ran } A, \langle \cdot, \cdot \rangle_A)$  is a pre-Hilbert space. We will denote by  $\mathcal{H}_A$  its completion and define

$$T : \mathcal{B} \longrightarrow \mathcal{H}_A, \quad Tb \stackrel{\text{def}}{=} Ab.$$

We have for  $b, c \in \mathcal{B}$

$$\langle T^*(Ac), b \rangle_{\mathcal{B}} = \langle Ac, Tb \rangle_{\mathcal{H}_A} = \langle Ac, b \rangle_{\mathcal{B}}.$$

Hence,  $T^*$  extends continuously to the injection map from  $\mathcal{H}_A$  into  $\mathcal{B}^*$ . We note that

$$\langle Tb, Tc \rangle_{\mathcal{H}_A} = \langle T^*Tb, c \rangle_{\mathcal{B}} = \langle Ab, c \rangle_{\mathcal{B}}$$

(that is, (2.3) holds). The claim on the norms is proved as follows: we have

$$\|A\| = \|T^*T\| \leq \|T\|^2$$

on the one hand and

$$\|A\| \geq \sup_{\|b\|=1} \langle Ab, b \rangle_{\mathcal{B}} = \sup_{\|b\|=1} \langle T^*Tb, b \rangle_{\mathcal{B}} = \sup_{\|b\|=1} \langle Tb, Tb \rangle_{\mathcal{H}_A} = \|T\|^2,$$

that is,  $\|A\| \geq \|T\|^2$  on the other hand. Combining the two inequalities we obtain (2.4). We now prove (2.5). We have:

$$\begin{aligned} |\langle Ab, c \rangle_{\mathcal{B}}| &= \langle Tb, Tc \rangle_{\mathcal{H}} \\ &\leq \langle Tb, Tb \rangle_{\mathcal{H}}^{1/2} \langle Tc, Tc \rangle_{\mathcal{H}}^{1/2} \\ &= \langle Ab, b \rangle_{\mathcal{B}} \langle Ac, c \rangle_{\mathcal{B}}. \end{aligned}$$

□

We will say that  $A \leq B$  if  $B - A \geq 0$ . Note that

$$(2.8) \quad A \leq B \implies \|A\| \leq \|B\|.$$

Indeed, from (2.4) we have:

$$\|A\| = \sup_{\|b\|=1} \langle Ab, b \rangle_{\mathcal{B}} \leq \sup_{\|b\|=1} \langle Bb, b \rangle_{\mathcal{B}} = \|B\|.$$

**Definition 2.3.** Let  $\mathcal{H}$  be a Hilbert space of  $\mathcal{B}^*$ -valued functions defined on a set  $\Omega$  and let  $K(z, w)$  be an  $\mathbf{L}(\mathcal{B}, \mathcal{B}^*)$ -valued kernel defined on  $\Omega \times \Omega$ . The kernel  $K(z, w)$  is called the reproducing kernel of the Hilbert space  $\mathcal{H}$  if for every  $w \in \Omega$  and  $b \in \mathcal{B}$   $K(\cdot, w)b \in \mathcal{H}$  and

$$\langle f, K(\cdot, w)b \rangle_{\mathcal{H}} = \langle f(w), b \rangle_{\mathcal{B}}, \quad \forall f \in \mathcal{H}.$$

**Definition 2.4.** Let  $K(z, w)$  be an  $\mathbf{L}(\mathcal{B}, \mathcal{B}^*)$ -valued kernel defined on  $\Omega \times \Omega$ . The kernel  $K(z, w)$  is said to be positive if for any choice of  $z_1, \dots, z_n \in \Omega$  and  $b_1, \dots, b_n \in \mathcal{B}$  it holds that

$$\sum_{j=1}^n \langle K(z_i, z_j) b_j, b_i \rangle_{\mathcal{B}} \geq 0.$$

**Proposition 2.5.** *The reproducing kernel  $K(z, w)$  of a Hilbert space of  $\mathcal{B}^*$ -valued functions, when it exists, is unique and positive.*

**Proposition 2.6.** *Let  $K(z, w)$  be an  $\mathbf{L}(\mathcal{B}, \mathcal{B}^*)$ -valued positive kernel defined on  $\Omega \times \Omega$ . Then there exists a unique Hilbert space of  $\mathcal{B}^*$ -valued functions defined on  $\Omega$  with the reproducing kernel  $K(z, w)$ .*

The proofs of these propositions are the same as in the Hilbert space case and are therefore omitted.

*Remark 2.7.* One can derive the notion of a reproducing kernel Hilbert space of  $\mathcal{B}$ -valued functions from Definition 2.3 above, using the natural injection  $\tau$  from  $\mathcal{B}$  into  $\mathcal{B}^{**}$  defined by (2.2).

### 3. STIELTJES INTEGRAL

In this section we define the Stieltjes integral of a scalar function with respect to a  $\mathbf{L}(\mathcal{B}, \mathcal{B}^*)$ -valued positive measure. We here follows the analysis presented in [10, §4 p. 19] for the case of operators in Hilbert spaces. We consider a separable Banach space  $\mathcal{B}$  and an increasing positive function

$$M : [a, b] \longrightarrow \mathbf{L}(\mathcal{B}, \mathcal{B}^*).$$

Thus,  $M(t) \geq 0$  for all  $t \in [a, b]$  and moreover

$$a \leq t_1 \leq t_2 \leq b \implies M(t_2) - M(t_1) \geq 0.$$

Let  $f(t)$  be a scalar continuous function on  $[a, b]$  and let

$$a = t_0 \leq \xi_1 \leq t_1 \leq \xi_2 \leq t_2 \leq \dots \leq \xi_m \leq t_m = b$$

be a subdivision of  $[a, b]$ . The Stieltjes integral  $\int_a^b f(t) dM(t)$  is defined to be the limit (in the  $\mathbf{L}(\mathcal{B}, \mathcal{B}^*)$  topology) of the sums of the form

$$\sum_{j=1}^m f(\xi_j) (M(t_j) - M(t_{j-1}))$$

as  $\sup_j |t_j - t_{j-1}|$  goes to 0.

**Theorem 3.1.** *The integral  $\int_a^b f(t) dM(t)$  exists.*

The proof of Theorem 3.1 is done along the lines of [10]. First we need the following lemma.

**Lemma 3.2.** *Let  $\alpha_j$  and  $\beta_j$  be complex numbers such that  $|\alpha_j| \leq |\beta_j|$  ( $j = 1, 2, \dots, m$ ). Let  $H_1, \dots, H_m$  be positive operators from  $\mathcal{B}$  into  $\mathcal{B}^*$ . Then it holds that*

$$\left\| \sum_{j=1}^m \alpha_j H_j \right\| \leq \left\| \sum_{j=1}^m |\beta_j| H_j \right\|.$$

*Proof.* For each  $j$  we write  $H_j = T_j^* T_j$  where  $T_j$  is a bounded operator from  $\mathcal{B}$  into some Hilbert space  $\mathcal{H}_j$  as in Theorem 2.2. Then for  $b, c \in \mathcal{B}$  of modulus 1 we have:

$$\begin{aligned}
\left| \left\langle \sum_{j=1}^m \alpha_j H_j b, c \right\rangle_{\mathcal{B}} \right| &= \left| \sum_{j=1}^m \langle \alpha_j H_j b, c \rangle_{\mathcal{B}} \right| \\
&\leq \sum_{j=1}^m |\alpha_j| \cdot |\langle T_j b, T_j c \rangle_{\mathcal{H}_j}| \\
&\leq \sum_{j=1}^m \sqrt{|\beta_j|} \|T_j b\|_{\mathcal{H}_j} \sqrt{|\beta_j|} \|T_j c\|_{\mathcal{H}_j} \\
&\leq \left( \sum_{j=1}^m |\beta_j| \|T_j b\|_{\mathcal{H}_j}^2 \right)^{1/2} \left( \sum_{j=1}^m |\beta_j| \|T_j c\|_{\mathcal{H}_j}^2 \right)^{1/2} \\
&\leq \left( \sum_{j=1}^m |\beta_j| \langle H_j b, b \rangle_{\mathcal{B}} \right)^{1/2} \left( \sum_{j=1}^m |\beta_j| \langle H_j c, c \rangle_{\mathcal{B}} \right)^{1/2} \\
&\leq \left\| \sum_{j=1}^m |\beta_j| H_j \right\|
\end{aligned}$$

where we have used (2.4) to get the last inequality. Thus, taking the supremum on  $c$  (of unit norm) we have  $\left\| \sum_{j=1}^m \alpha_j H_j b \right\| \leq \left\| \sum_{j=1}^m |\beta_j| H_j \right\|$ , and hence the required inequality.  $\square$

*Proof of Theorem 3.1.* It suffices to show that for every  $\epsilon > 0$  there exists  $\delta > 0$  such that if

$$(3.1) \quad a = t_0 \leq t_1 \leq \dots \leq t_m = b$$

is a subdivision of  $[a, b]$  such that  $\max_j |t_j - t_{j-1}| \leq \delta$  and

$$a = t_0 = t_1^0 \leq \dots \leq t_1^{k_1} = t_1 = t_2^0 \leq \dots \leq t_m^{k_m} = t_m = b$$

is a continuation of the subdivision (3.1) then for every choice of  $\xi_j \in [t_{j-1}, t_j]$  and  $\xi_j^\ell \in [t_j^{\ell-1}, t_j^\ell]$  it holds that

$$\left\| \sum_{j=1}^m f(\xi_j) (M(t_j) - M(t_{j-1})) - \sum_{j=1}^m \sum_{\ell=1}^{k_j} f(\xi_j^\ell) (M(t_j^\ell) - M(t_j^{\ell-1})) \right\| \leq \epsilon.$$

Let us take  $\delta$  such that

$$|t' - t''| \leq \delta \implies |f(t') - f(t'')| \leq \frac{\epsilon}{\|M(b) - M(a)\|}.$$

Then the desired conclusion follows from Lemma 3.2.  $\square$

#### 4. HELLY'S THEOREM

The following theorem is proved in the case of *separable* Hilbert space in [10] (see Theorem 4.4 p. 22 there). The proof goes in the same way in the case of separable Banach spaces. We quote it in a version adapted to the present setting.

**Theorem 4.1.** *Let  $F_n(t)$  ( $t \in [0, 2\pi]$ ) be a sequence of positive increasing  $\mathbf{L}(\mathcal{B}, \mathcal{B}^*)$ -valued functions such that*

$$F_n(t) \leq F_0, \quad n = 0, 1, \dots \quad \text{and} \quad t \in [0, 2\pi],$$

*where  $F_0$  is some bounded positive operator. Then, there exists a subsequence of  $F_n$  (which we still denote by  $F_n$ ) which converges weakly for every  $t \in [0, 2\pi]$ . Moreover, for  $f(t)$  a continuous scalar function we have (in the weak sense, and via the subsequence):*

$$\int_0^{2\pi} f(t) dF(t) = \lim_{n \rightarrow \infty} \int_0^{2\pi} f(t) dF_n(t).$$

The proof given in [10] relies on the hypothesis of separability and on the inequalities

$$(4.1) \quad \begin{aligned} |\langle F_n(t)x, y \rangle_{\mathcal{B}}| &\leq \|F_0\| \cdot \|x\| \cdot \|y\|, \quad x, y \in \mathcal{B} \\ \sum_{\ell=1}^m |\langle \Delta_{\ell,n} Fx, y \rangle_{\mathcal{B}}| &\leq 2\|F_0\| \cdot \|x\| \cdot \|y\| \end{aligned}$$

where  $0 = t_0 \leq t_1 < \dots < t_m = 2\pi$  and  $\Delta_{\ell,n} F = F_n(t_\ell) - F_n(t_{\ell-1})$ . The first inequality follows from (2.8). We prove the second one using the factorization given in Theorem 2.2. Using this theorem we write:

$$\Delta_{\ell,n} = T_{\ell,n}^* T_{\ell,n}$$

where  $T_{\ell,n}$  is a bounded operator from some Hilbert space  $\mathcal{H}_{\ell,n}$  into  $\mathcal{B}$ . Then, using (2.5) we have:

$$\begin{aligned} \sum_{\ell=1}^m |\langle \Delta_{\ell,n} Fx, y \rangle_{\mathcal{B}}| &\leq \sum_{\ell=1}^m \langle \Delta_{\ell,n} Fx, x \rangle_{\mathcal{B}}^{1/2} \langle \Delta_{\ell,n} Fy, y \rangle_{\mathcal{B}}^{1/2} \\ &\leq \left( \sum_{\ell=1}^m \langle \Delta_{\ell,n} Fx, x \rangle_{\mathcal{B}} \right)^{1/2} \left( \sum_{\ell=1}^m \langle \Delta_{\ell,n} Fy, y \rangle_{\mathcal{B}} \right)^{1/2} \\ &= \langle (F(2\pi) - F(0))x, x \rangle_{\mathcal{B}}^{1/2} \langle (F(2\pi) - F(0))y, y \rangle_{\mathcal{B}}^{1/2} \\ &\leq \langle 2F_0 x, x \rangle_{\mathcal{B}}^{1/2} \langle 2F_0 y, y \rangle_{\mathcal{B}}^{1/2} \\ &\leq 2\|F_0\| \cdot \|x\| \cdot \|y\|. \end{aligned}$$

The proof then proceeds as follows (see [10, p. 22]). One applies the first inequality in (4.1) for  $x, y$  in a dense countable set  $E$  of  $\mathcal{B}$ . The functions  $t \mapsto \langle F_n(t)x, y \rangle_{\mathcal{B}}$  are of bounded variation. An application of the scalar case of Helly's theorem and the diagonal process allows to find a subsequence of  $F_n$  such that for all  $x, y \in E$  and every  $t \in [0, 2\pi]$  the limit

$$\lim_{n \rightarrow \infty} \langle F_n(t)x, y \rangle_{\mathcal{B}}$$

exists. We refer the reader to [10] for more details.

## 5. $\mathbf{L}(\mathcal{B}, \mathcal{B}^*)$ -VALUED CARATHÉODORY FUNCTIONS

**Definition 5.1.** Let  $\phi(z)$  be an  $\mathbf{L}(\mathcal{B}, \mathcal{B}^*)$ -valued function, weakly continuous at the origin in the sense that

$$(5.1) \quad \langle \phi(z)b, b \rangle_{\mathcal{B}} \rightarrow \langle \phi(0)b, b \rangle_{\mathcal{B}} \text{ as } z \rightarrow 0, \quad \forall b \in \mathcal{B}.$$

For a Carathéodory function  $\phi(z)$  we shall denote by  $\mathcal{L}(\phi)$  the Hilbert space of  $\mathcal{B}^*$ -valued functions with the reproducing kernel  $k_\phi(z, w)$ .

We give two representation theorems for Carathéodory functions. In the first we make no assumption on the space  $\mathcal{B}$ . Following arguments of Krein and Langer (see [17]), we prove the existence of a realization of the form (1.3). The second theorem assumes that the space  $\mathcal{B}$  is separable. We prove that in this case the Carathéodory functions can be characterized as functions analytic in the open unit disk with positive real part. Then we derive a Herglotz-type representation formula.

**Theorem 5.2.** *Let  $\Omega$  be a neighborhood of the origin and let  $\phi(z)$  be an  $\mathbf{L}(\mathcal{B}, \mathcal{B}^*)$ -valued function defined in  $\Omega$  and weakly continuous at the origin (in the sense (5.1)). Then  $\phi(z)$  is a Carathéodory function if and only if it admits the representation*

$$\phi(z)^*|_{\mathcal{B}} = D + C^*(I - z^*V)^{-1}(I + z^*V)C, \quad z \in \Omega,$$

or equivalently,

$$(5.2) \quad \phi(z) = D^*|_{\mathcal{B}} + C^*(I + zV^*)(I - zV^*)^{-1}C, \quad z \in \Omega,$$

where  $V$  is an isometric operator in some Hilbert space  $\mathcal{H}$ ,  $C$  is a bounded operator from  $\mathcal{B}$  into  $\mathcal{H}$  and  $D$  is a purely imaginary operator from  $\mathcal{B}$  into  $\mathcal{B}^*$  in the sense that

$$(5.3) \quad D + D^*|_{\mathcal{B}} = 0.$$

In particular, every Carathéodory function has an analytic extension to the whole open unit disk.

*Proof.* Let  $\phi(z)$  be a Carathéodory function. First we observe that elements of  $\mathcal{L}(\phi)$  are weakly continuous at the origin:

$$\langle f(w), b \rangle_{\mathcal{B}} \rightarrow \langle f(0), b \rangle_{\mathcal{B}} \text{ as } w \rightarrow 0, \quad \forall f \in \mathcal{L}(\phi), b \in \mathcal{B}.$$

Indeed, this is a consequence of the Cauchy – Schwarz inequality as

$$\langle f(w), b \rangle_{\mathcal{B}} - \langle f(0), b \rangle_{\mathcal{B}} = \langle f, (k_\phi(\cdot, w) - k_\phi(\cdot, 0))b \rangle_{\mathcal{L}(\phi)}$$

and

$$\|(k_\phi(\cdot, w) - k_\phi(\cdot, 0))b\|_{\mathcal{L}(\phi)}^2 = \frac{|w|^2}{1 - |w|^2} \Re \langle \phi(w)b, b \rangle_{\mathcal{B}}.$$

We consider in  $\mathcal{L}(\phi) \times \mathcal{L}(\phi)$  the linear relation  $R$  spanned by the pairs

$$R = \left( \sum k_\phi(z, w_i)w_i^*b_i, \sum k_\phi(z, w_i)b_i - k_\phi(z, 0)(\sum b_i) \right)$$

where all the  $b_i \in \mathcal{B}$ , the  $w_i \in \Omega$  and where all the sums are finite. This relation is densely defined because of the weak continuity of the elements of  $\mathcal{L}(\phi)$  at the origin. Indeed, let  $f \in \mathcal{L}(\phi)$  be orthogonal to the domain of  $R$ . Then,

$$\langle f, k_\phi(\cdot, w)b \rangle_{\mathcal{L}(\phi)} = \langle f(w), b \rangle_{\mathcal{B}} = 0$$

for all  $b \in \mathcal{B}$  and all points  $w \neq 0$  in the domain of  $f$ . Thus  $f(w) = 0$  at all these points  $w$  and the continuity hypothesis implies that also  $f(0) = 0$ . The relation  $R$  is readily seen to be isometric. Its closure is thus the graph of an isometry, which we call  $V$ . We have:

$$V(k_\phi(z, w)w^*b) = k_\phi(z, w)b - k_\phi(z, 0)b,$$

and in particular

$$(5.4) \quad (I - w^*V)^{-1}k_\phi(\cdot, 0)b = k_\phi(\cdot, w)b.$$

Denote by  $C$  the map

$$C : \mathcal{B} \longrightarrow \mathcal{L}(\phi), \quad (Cb)(z) \stackrel{\text{def}}{=} k_\phi(z, 0)b.$$

Then, for  $f \in \mathcal{L}(\phi)$ ,  $C^*f = f(0)$ . Applying  $C$  on the left on both sides of (5.4) we obtain

$$\frac{\phi(0) + \phi(w)^*|_{\mathcal{B}}}{2}b = C^*(I - w^*V)^{-1}Cb.$$

Since

$$C^*Cb = \frac{\phi(0) + \phi(0)^*|_{\mathcal{B}}}{2}b$$

we obtain

$$\begin{aligned} \phi(0) + \phi(w)^*|_{\mathcal{B}} &= 2C^*(I - w^*V)^{-1}C - C^*C + C^*C \\ &= C^*(I - w^*V)^{-1}(I + w^*V)C + C^*C \end{aligned}$$

so that

$$\phi(w)^*|_{\mathcal{B}} + \frac{\phi(0) - \phi(0)^*|_{\mathcal{B}}}{2} = C^*(I - w^*V)^{-1}(I + w^*V)C,$$

which gives the required formula with

$$(5.5) \quad D = \frac{\phi(0) - \phi(0)^*|_{\mathcal{B}}}{2}.$$

We now prove the converse statement and first compute

$$\langle k_\phi(z, w)x, y \rangle_{\mathcal{B}} \quad \text{for } x, y \in \mathcal{B}.$$

We have

$$\langle \phi(w)^*|_{\mathcal{B}}x, y \rangle_{\mathcal{B}} = \langle Dx, y \rangle_{\mathcal{B}} + \langle (I - w^*V)^{-1}(I + w^*V)Cx, Cy \rangle_{\mathcal{L}(\phi)}.$$

We have, with  $\tau$  the natural injection from  $\mathcal{B}$  into  $\mathcal{B}^{**}$  (see (2.2)):

$$\begin{aligned} \langle \phi(z)x, y \rangle_{\mathcal{B}} &= \overline{\langle \tau y, \phi(z)x \rangle_{\mathcal{B}^*}} \\ &= \overline{\langle \phi(z)^*\tau y, x \rangle_{\mathcal{B}^*}} \\ (5.6) \quad &= \overline{\langle \phi(z)^*|_{\mathcal{B}}y, x \rangle_{\mathcal{B}}} \\ &= \overline{\langle Dy, x \rangle_{\mathcal{B}}} + \langle (I - zV^*)^{-1}(I + zV^*)Cx, Cy \rangle_{\mathcal{L}(\phi)} \\ &= \langle D^*|_{\mathcal{B}}x, y \rangle_{\mathcal{B}} + \langle (I - zV^*)^{-1}(I + zV^*)Cx, Cy \rangle_{\mathcal{L}(\phi)}, \end{aligned}$$

and so

$$\langle k_\phi(z, w)x, y \rangle_{\mathcal{B}} = \langle (I - zV^*)^{-1}Cx, (I - wV^*)^{-1}Cy \rangle_{\mathcal{L}(\phi)},$$

from which follows the positivity of  $k_\phi(z, w)$ . Finally, (5.6) also implies (5.2) and this concludes the proof.  $\square$

*Remark 5.3.* Although the above argument is very close to the one in [17, p. 365–366] we note the following: we use a concrete space (the space  $\mathcal{L}(\phi)$ ) to build the relation rather than abstract elements and the relation  $R$  is defined slightly differently.

*Remark 5.4.* As already mentioned, the above argument still goes through when the kernel has a finite number of negative squares. In this case the space  $\mathcal{L}(\phi)$  is a Pontryagin space. For  $b \in \mathcal{B}$  and sufficiently small  $h \in \mathbb{C}$  we consider the functions  $f_h(z) = (K(z, w + h) - K(z, w))b$ , which have the following properties:

$$\begin{aligned} \lim_{h \rightarrow 0} \langle f_h, f_h \rangle_{\mathcal{L}(\phi)} &= 0, \\ \lim_{h \rightarrow 0} \langle f, f_h \rangle_{\mathcal{L}(\phi)} &= 0, \quad \forall f \in \text{span}\{k_\phi(\cdot, w)b\}. \end{aligned}$$

It follows from the convergence criteria in Pontryagin spaces (see [16], [17, p. 356]) that

$$\lim_{h \rightarrow 0} \langle f, f_h \rangle_{\mathcal{L}(\phi)} = 0, \quad \forall f \in \mathcal{L}(\phi).$$

The fact that the relation is the graph of an isometric operator is proved in [3, Theorem 1.4.2 p. 29]. This follows from a theorem of Shmulyan which states that a contractive relation between Pontryagin spaces of same index is the graph of a contractive operator; see [23] and [3, Theorem 1.4.1 p. 27].

**Theorem 5.5.** *Let  $\mathcal{B}$  be a separable Banach space and let  $\phi(z)$  be a  $\mathbf{L}(\mathcal{B}, \mathcal{B}^*)$ -valued function analytic in the open unit disk, such that*

$$\phi(z) + \phi(z)^* \big|_{\mathcal{B}} \geq 0.$$

*Then there exists an increasing  $\mathbf{L}(\mathcal{B}, \mathcal{B}^*)$ -valued function  $M(t)$  ( $t \in [0, 2\pi]$ ) and a purely imaginary operator  $D$  (that is, satisfying (5.3)) such that*

$$\phi(z) = D + \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} dM(t),$$

*where the integral is defined in the weak sense. Furthermore the kernel  $k_\phi(z, w)$  is positive in  $\mathbb{D}$ .*

*Proof.* We follow the arguments in [10], and will apply Theorem 4.1. The separability hypothesis of  $\mathcal{B}$  is used at this point.

We first assume that  $\phi(z)$  is analytic in  $|z| < 1 + \epsilon$  with  $\epsilon > 0$ . We have (the existence of the integrals follows from Theorem 3.1):

$$\begin{aligned} \frac{1}{4\pi} \int_0^{2\pi} \phi(e^{it}) \frac{e^{it} + z}{e^{it} - z} dt &= \phi(z) - \frac{\phi(0)}{2} \\ \frac{1}{4\pi} \int_0^{2\pi} (\phi(e^{it})^*) \big|_{\mathcal{B}} \frac{e^{it} + z}{e^{it} - z} dt &= \frac{1}{2\pi} \int_0^{2\pi} (\phi(e^{-it})^*) \big|_{\mathcal{B}} \frac{e^{-it} + z}{e^{-it} - z} dt \\ &= \frac{(\phi(0))^* \big|_{\mathcal{B}}}{2}. \end{aligned}$$

Thus,

$$\phi(z) = D + \int_0^{2\pi} \frac{(\phi(e^{it}) + (\phi(e^{it}))^* \big|_{\mathcal{B}})}{4\pi} \frac{e^{it} + z}{e^{it} - z} dt,$$

with  $D$  as in (5.5) and the formula for general  $\phi(z)$  follows from Helly's theorem applied to the measures

$$\frac{(\phi(re^{it}) + (\phi(re^{it}))^* \big|_{\mathcal{B}})}{4\pi} dt, \quad r < 1$$

(or more precisely to a sequence  $r_n \rightarrow 1$ ).

We now prove the positivity of the kernel  $k_\phi(z, w)$  and first assume that  $\phi(z)$  is analytic in  $|z| < 1 + \epsilon$  as above. We have:

$$\frac{1}{4\pi} \int_0^{2\pi} (\phi(e^{-it}))^*|_{\mathcal{B}} \frac{e^{it} + z}{e^{it} - z} dt = (\phi(z^*))^*|_{\mathcal{B}} - \frac{(\phi(0))^*|_{\mathcal{B}}}{2}.$$

Thus

$$(\phi(z^*))^*|_{\mathcal{B}} - \frac{(\phi(0))^*|_{\mathcal{B}}}{2} = \frac{1}{4\pi} \int_0^{2\pi} (\phi(e^{it}))^*|_{\mathcal{B}} \frac{e^{-it} + z}{e^{-it} - z} dt$$

and so

$$(\phi(z^*))^*|_{\mathcal{B}} = -D + \int_0^{2\pi} \frac{(\phi(e^{it}) + (\phi(e^{it}))^*|_{\mathcal{B}})}{4\pi} \frac{1 + ze^{it}}{1 - ze^{it}} dt$$

since

$$\frac{1}{4\pi} \int_0^{2\pi} \phi(e^{it}) \frac{1 + ze^{it}}{1 - ze^{it}} dt = \frac{1}{4\pi} \int_0^{2\pi} \phi(e^{-it}) \frac{e^{it} + z}{e^{it} - z} dt = \frac{\phi(0)}{2}.$$

Thus,

$$k_\phi(z, w) = \int_0^{2\pi} \frac{(\phi(e^{it}) + (\phi(e^{it}))^*|_{\mathcal{B}})}{4\pi} \frac{1}{(e^{it} - z)(e^{it} - w)^*} dt.$$

The positivity follows. The case of general  $\phi(z)$  is done by approximation using Helly's theorem; indeed using Theorem 4.1 we have for general  $\phi(z)$ :

$$\begin{aligned} \langle k_\phi(w_\ell, w_j) b_j, b_\ell \rangle_{\mathcal{B}} &= \\ &= \lim_{r \rightarrow 1} \int_0^{2\pi} \left\langle \frac{(\phi(re^{it}) + (\phi(re^{it}))^*|_{\mathcal{B}})}{4\pi} \frac{1}{(e^{it} - w_\ell)(e^{it} - w_j)^*} b_j, b_\ell \right\rangle_{\mathcal{B}} dt \\ &= \lim_{r \rightarrow 1} \int_0^{2\pi} \left\langle \frac{(\phi(re^{it}) + (\phi(re^{it}))^*|_{\mathcal{B}})}{4\pi} \frac{b_j}{(e^{it} - w_j)^*}, \frac{b_\ell}{(e^{it} - w_\ell)^*} \right\rangle_{\mathcal{B}} dt. \end{aligned}$$

□

## 6. THE CASE OF $\mathbf{L}(\mathcal{B}^*, \mathcal{B})$ -VALUED FUNCTIONS

We turn to the case of  $\mathbf{L}(\mathcal{B}^*, \mathcal{B})$ -valued functions. Using the natural injection  $\tau$

$$\mathcal{B} \xrightarrow{\tau} \mathcal{B}^{**}$$

defined by (2.2) we shall say that an  $\mathbf{L}(\mathcal{B}^*, \mathcal{B})$ -valued function  $\phi(z)$  is a Carathéodory function if the  $\mathbf{L}(\mathcal{B}^*, \mathcal{B}^{**})$ -valued function  $\tau\phi(z)$  is a Carathéodory function.

**Theorem 6.1.** *An  $\mathbf{L}(\mathcal{B}^*, \mathcal{B})$ -valued function  $\phi(z)$  defined in a neighborhood of the origin and weakly continuous at the origin is a Carathéodory function if and only if it admits the representation*

$$\phi(z)^* = D + C^*(I - z^*V)^{-1}(I + z^*V)C,$$

or, equivalently,

$$\tau\phi(z) = D^*|_{\mathcal{B}_*} + C^*(I + zV^*)(I - zV^*)^{-1}C,$$

where  $V$  is an isometric operator in some Hilbert space  $\mathcal{H}$ , where  $C$  is a bounded operator from  $\mathcal{B}^*$  into  $\mathcal{H}$  and where  $D$  is a purely imaginary operator from  $\mathcal{B}^*$  into  $\mathcal{B}^{**}$ .

In particular  $\phi(z)$  has an analytic extension to the whole open unit disk.

*Proof.* By Theorem 5.2 (with  $\mathcal{B}$  replaced by  $\mathcal{B}^*$ ),  $\phi(z)$  is a Carathéodory function if and only if

$$(\tau\phi(z))^* \big|_{\mathcal{B}^*} = D + C^*(I - z^*V)^{-1}(I + z^*V)C,$$

where  $C, V, D$  have the stated properties. But  $(\tau\phi(z))^* \big|_{\mathcal{B}^*} = \phi(z)^*$ .  $\square$

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